Peter-Weyl Theorem

Yiyang Liu
liuyiy@sas.upenn.edu

Niuniu Zhang
niuniu@sas.upenn.edu

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1 Preliminary

This presentation assumes the basic familiarity of L^p-spaces, inner product spaces, Hilbert spaces, definition of topological groups, and the very basics of measure theory.

1.1 The L^p-spaces

Definition 1.1 (rudin, 65)**.** Suppose X is any arbitrary measure space with measure μ. If $0 < p < \infty$ and if f is a complex measurable function on X, define

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}$$

and let $L^p(X)$ consist of all f for which

$$\|f\|_p < \infty$$

We call $\|f\|_p$ the L^p-norm of f.

1.2 Topological Groups

Definition 1.2 (gallier, 249)**.** A group G with identity 1 is a topological group if G is a Hausdorff topological space and the map

$$G \times G \to G : (g, h) \mapsto gh^{-1}$$

is continuous.

Definition 1.3 (gallier, 252)**.** A topological space X is locally compact if and only if for every point $p \in X$, there is a compact neighborhood C of p; that is, there is a compact C and an open U, with $p \in U \subset C$. For example, the additive group $(\mathbb{R}, +)$ is locally compact.

Definition 1.4 (gallier, 257)**.** Let G be a topological group and X be a topological space. G acts continuously on X if the map $\varphi : G \times X \to X$ is continuous.

If φ is continuous, then each map $\varphi_g : X \to X$ is a homeomorphism with $\varphi_g(x) = g \cdot x$ for all $x \in X$.

1.3 Haar Measure

G is an extremely general class of objects, including familiar structures such as $\mathbb{R}$, but certainly also including much more structures that may not be as nice. To define integral over functions whose domains are $\mathbb{R}$, we can use the Lebesgue measure. Therefore, to define integral over functions whose domains are these very general G, we also need some kind of measure.

Theorem 1.1 (Bergen, 2)**.** Every locally compact group G admits a positive Borel measure μ, called the Haar measure, where

$$\mu(g \cdot X) = \mu(X) = \mu(X \cdot g)$$

for all $g \in G$ and μ-measurable sets X.

At this point, we may introduce the Hilbert Space $L^2(G)$, space of square-integrable functions defined on G, and its inner product is given by

$$\langle f_1, f_2 \rangle_{L^2} = \int_G f_1(g)\overline{f_2(g)} d\mu(g)$$

1.4 Hilbert Space

Loosely speaking, Hilbert Space is a normed vector space, where Cauchy sequence converges with respect to the norm. Having established the norm and inner product of $L^2(G)$, we introduce some important propositions and corollaries regarding the L^p-space and facts about Hilbert Space.

Proposition 1.1 (Bergen, 18)**.** For $1 \leq p < \infty$, $\text{Fun}(G, \mathbb{C})$ is dense in $L^p(G)$. In particular, if M, a collection of functions, is dense in $\text{Fun}(G, \mathbb{C})$, then M is dense in $L^2(G)$.

We leave the proof as an exercise. Now, we dive into the discussion of Hilbert Space.

Definition 1.5 (Compact Operator)**.** Suppose H is a Hilbert Space with the standard inner product and norm. Take $\mathcal{B}$ to be a closed unit ball. A linear operator $T : H \to H$ is Compact if $\overline{T(\mathcal{B})}$ is compact.

Definition 1.6 (Self-adjoint Operator)**.** Recall from linear algebra that a Hermitian Matrix M is precisely defined as $M = M^* = \overline{M^T}$. We defined a linear operator $T : H \to H$ to be self-adjoint if $T = T^*$, that is, for all $x, y \in H$, $\langle Tx, y \rangle = \langle x, Ty \rangle$

Recall from linear algebra that spectral decomposition deals with the orthogonal diagonalizability. In a similar vein, we introduce the Spectral Theorem for Hilbert Space and lead to the decomposition of a Hilbert Space.

Theorem 1.2 (Spectral Theorem)**.** Suppose T is a compact self-adjoint operator on H and $\{\lambda_i\}_{i \in I}$ are distinct eigenvalues of T that are associated with eigenspaces $\{M_{\lambda_i}\}_{i \in I}$, then we may decompose H as an orthogonal Hilbert Space direct sum, i.e,

$$H = \bigoplus_{i \in I} M_{\lambda_i}$$

The ring multiplication of $\mathbb{C}[G]$ coincides with the notion of convolution operator.

Definition 1.7. Take ϕ to be continuous complex function defined on G, then we define the convolution operator T_ϕ by

$$T_\phi f(x) = \int_G \phi(xg^{-1}) f(g) dg$$

note that T_ϕ is a bounded, compact, and self-adjoint operator.

Proposition 1.2. For $\lambda \in \mathbb{C}$, let M_λ be the corresponding eigenspace for T_ϕ. For each $g \in G$, we define

$$f_g(x) = f(xg)$$

then for each $f \in M_\lambda, f_g \in M_\lambda$.

2 Representation Theory

As the Peter-Weyl Theorem is all about decomposing $L^2(G)$, the square integrable functions defined on G, we construct a subspace $\mathcal{M}$, where the elements are called matrix coefficients. Then, we show $\mathcal{M}$ is dense in $L^2(G)$.
Recall two important theorems from Representation Theory, namely Mashke's Theorem and Schur's Orthogonality Relations.

Theorem 2.1 (Orthogonality Relations)**.** Let V_ρ and V_π be two nonisomorphic irreducible representations of G with respective G-invariant inner products $\langle -,-\rangle_\rho, \langle -,-\rangle_\pi$. Then for all $a, v \in V_\rho, b, w \in V_\pi$

$$\int_G \overline{\langle ga, v\rangle_\rho}\langle gb, w\rangle_\pi dg = 0$$

Theorem 2.2 (Mashke)**.** Let V be a representation of the compact group G. If U is a subrepresentation of V, then there exists a subrepresentation W of V such that $V = U \oplus W$.

In particular, the orthogonality relations establish the orthogonality of the Matrix coefficient.

3 Theorem

In this presentation, we are primarily interested in finding representation of the topological group G in the vector space $L^2(G)$. However, in the case where G is finite, we have actually already seen in our homework how such a representation work.

Recall the regular representation $\mathbb{C}[G]$, where the vector space consists of formal expressions of the form

$$\sum_{g\in G} \lambda_g g$$

and acting by $h \in G$ just amounts to a left translation:

$$\rho(h)\left(\sum_{g\in G} \lambda_g g\right) = \sum_{g\in G} \lambda_g hg$$

In Homework 2, we have seen how this is isomorphic to the space $L^2(G)$, which for us was just $\text{Fun}(G, \mathbb{C})$. However, in the case where G is infinite, we run into problems of continuity, and we no longer have $\mathbb{C}[G]$ being isomorphic to $L^2(G)$.

Definition 3.1. Given a finite dimensional representation $\rho : G \to GL(V_\rho)$, and some G-invariant inner product on V_ρ, we can define the corresponding **matrix coefficient**, which is just a function $\sigma_{\rho,v_1,v_2} : G \to \mathbb{C}$ such that $\sigma_{\rho,v_1,v_2}(g) = \langle \rho(g)v_1, v_2\rangle_\rho$ for some $v_1, v_2 \in V_\rho$

Proposition 3.1. Denote $\mathcal{M}$ as the set of matrix coefficients of G. $\mathcal{M}$ is closed under pointwise addition and scalar multiplication.

Definition 3.2. For a particular irreducible representation $\rho : G \to GL(V_\rho)$, we can define the subspace

$$\mathcal{M}_\rho = Span\{\sigma_{\rho,v_1,v_2} \in \mathcal{M} : v_1, v_2 \in V_\rho\}$$

which is just the span of all of the matrix coefficients associated to the irreducible representation ρ. An element of $\mathcal{M}$ is called a **matrix coefficient of the representation** ρ. In particular, M is closed under pointwise addition and scalar multiplication.

Proposition 3.2. Let $[\rho]$ denote the an equivalence class of isomorphic representations of G. If $\pi \in [\rho]$, then we know $\mathcal{M}_\pi = \mathcal{M}_\rho$.

At this point, from Mashke's Theorem and the closure property of $\mathcal{M}$, we know a matrix coefficient of a reducible representation is a sum of matrix coefficients of irreducible representations, i.e,

$$\mathcal{M} = \text{Span}\{\bigcup_{[\rho]} \mathcal{M}_{[\rho]}\}$$

By Schur's Orthogonality relation, we have

$$\mathcal{M} = \bigoplus_{[\rho]} \mathcal{M}_{[\rho]}$$

Proposition 3.3. Let $V_\rho \otimes V_\rho^*$ be the representation of G under the aciton

$$g(v_1 \otimes \langle -, v_2 \rangle_\rho) = gv_1 \otimes \langle -, v_2 \rangle_\rho$$

then

$$\mathcal{M} \cong \bigoplus_{[\rho]} V_\rho \otimes V_\rho^* \cong \bigoplus_{[\rho]} V_\rho^{\dim V_\rho}$$

Having constructed the matrix coefficient $\mathcal{M}$, we show $\mathcal{M}$ is dense in $L^2(G)$.

Definition 3.3. Given any $f : G \to \mathbb{C}$, we could define

$$\begin{aligned} f_g : G &\longrightarrow \mathbb{C} \\ x &\longmapsto f(xg) \end{aligned}$$

f is called **right G-finite** if the set $\{f_g : g \in G\}$ spans a finite dimensional vector space.

Proposition 3.4. Given any $f : G \to \mathbb{C}$, f is a matrix coefficient if and only if it is right G-finite.

Theorem 3.1. The set of matrix coefficients of G is dense in $C(G, \mathbb{C})$ (the subspace of continuous functions)

Theorem 3.2. (Peter-Weyl Theorem Part I). As a representation of G,

$$L^2(G) \cong \overline{\mathcal{M}} \cong \overline{\bigoplus_{[\rho]} V_\rho^{\dim V_\rho}} = \widehat{\bigoplus_{[\rho]}} V_\rho^{\dim V_\rho}$$

To be clear, we are summing over all irreducible representation class of G.