Math 602 Final Review Sheet

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1. **Abelian groups**
   - Every finitely generated abelian group G is isomorphic to a finite direct sum of cyclic groups, each of which is either infinite or of order a power of a prime.
   - $|G| = n \implies \exists H \subset G, |H| = m, \forall m|n$
2. **Sylow Theorem**
   - (Cauchy): If G finite group, $p||G|$, then $\exists g \in G$ with $|g| = p$.
   - (p-groups): G is a p-groups iff $|G| = p^n$.
   - The center of p-group is non-trivial.
   - (First Sylow): $|G| = p^n m$, with $(p, m) = 1$, then G contains a Sylow p-subgroup.
     - $|H| = p^n \iff H$ Sylow
     - $\exists g \in G, gHg^{-1}$ Sylow
     - Only one Sylow P, then $P \trianglelefteq G$
   - (Second Sylow): If H is a p-subgroup of G, and P Sylow, then there exists $x \in G$ such that $H \subset xPx^{-1}$. Any two Sylow are conjugate.
   - (Third Sylow): Number of Sylow p-subgroups, say s, $s||G|$, and $s = kp + 1$.
3. **Classification**
   - p, q prime with $p > q$. If $q \nmid p - 1$, then for all $|G| = pq$, we have $G \cong C_{pq}$. If $q|p - 1$, then either $G \cong C_{pq}$ or $G \cong K$, non abelian generated by $c, d \in K$ that
     $$|c| = p, |d| = q; \quad dc = c^s d, s \not\equiv \mod p, s^q \equiv 1 \mod p$$
   - p odd prime, then $|G| = 2p$ is isomorphic either to $\mathbb{Z}_{2p}$ or D_p.
   - Every group of order p^2 is abelian.
   - A group of order p^2 is either cyclic, or the product of two cyclic group of order p.
   - For every $n \geq 5$, the alternating group A_n is a simple group.
   - $|G| = 8 \implies G \cong Q_8$ or D_4.
   - Every group of order 15 is Cyclic.
   - $|G| = 6 \implies G \cong C_6, G \cong S_3 \cong D_3$.
4. **Nilpotent**: A group G is nilpotent if $Z_n(G) = G$ for some n.
   - $Z(G)$ is the center of G, so that $Z(G) \trianglelefteq G$.
   - $Z_0(G) = 1, Z_1(G) = G, Z_2(G) = Z(G/Z(G)), \ldots, Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$
   - $Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots \leq Z_c(G) = G$
   - Every p-group is nilpotent $\implies$ Every Abelian group is nilpotent

- The $\oplus$ of a finite number of nilpotent groups is nilpotent.
- G nilpotent iff it is the direct product of its Sylow subgroups.
- $Z(D_n) = 1$ if n odd, equals $\{1, r^{\frac{n}{2}}\}$ otherwise.

5. **Solvable**: A group G is solvable if G has a series of subgroups

$$1 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_k = G$$

where, for each $0 \leq i < k, H_i \trianglelefteq H_{i+1}$ and H_{i+1}/H_i Abelian.

- Every nilpotent group is solvable.
- S_n is not solvable for $n \geq 5$.
- (Feit-Thompson): Every finite G of odd order is solvable.
- (Burnside): If $|G| = p^a q^b$, then G is solvable.
- $|G| = pqr$ and p may equals q, then G solvable.
- (Hall): For all $p, |G| = p^a m$, G has a subgroup of order $m \implies G$ solvable.

6. **Short Exact Sequence**:

$$1 \to N \to G \to H \to 1 \quad H \cong G/N$$

is called short exact sequence. G is an extension of H by N.

- **Semi**